

Null-orbit reflexive operators

Don Hadwin
University of New Hampshire

Ileana Ionaşcu
Philadelphia University

Hassan Yousefi
California State University Fullerton

January 13, 2011

Abstract

We introduce and study the notion of null-orbit reflexivity, which is a slight perturbation of the notion of orbit-reflexivity. Positive results for orbit reflexivity and the recent notion of \mathbb{C} -orbit reflexivity both extend to null-orbit reflexivity. Of the two known examples of operators that are not orbit-reflexive, one is null-orbit reflexive and the other is not. The class of null-orbit reflexive operators includes the classes of hyponormal, algebraic, compact, strictly block-upper (lower) triangular operators, and operators whose spectral radius is not 1. We also prove that every polynomially bounded operator on a Hilbert space is both orbit-reflexive and null-orbit reflexive.

1 Introduction

In a recent paper [11] the authors and M. McHugh introduced a new notion of reflexivity for operators, *\mathbb{C} -orbit reflexivity* as well as its linear-algebraic analogue. This notion is related to the notion of orbit reflexivity [5]. Examples of Hilbert space operators that are not orbit reflexive can be found in two very remarkable papers; the first example was given by S. Grivaux and M. Roginskaya [1], and the second, much simpler, example was given by V. Müller and J. Vršovský [11].

Although even in finite-dimensions there is an ample supply of operators that are not \mathbb{C} -orbit reflexive, it was easy to show that operators that are strictly block-upper(or lower)-triangular are \mathbb{C} -orbit reflexive. This fact combined with the example of a non-orbit-reflexive operator in [11], led us naturally to a new version of orbit reflexivity, null-orbit reflexivity, that includes all of the previously-proved orbit-reflexive operators but excludes the counterexample in [11].

Suppose T is a linear transformation on a vector space. We define the *null-orbit* of T as

$$\text{nullOrb}(T) = \{0, 1, T, T^2, \dots\}.$$

The *orbit* of T is $\text{Orb}(T) = \{1, T, T^2, \dots\}$. We define $\text{nullOrbRef}_0(T)$ to be the set of all linear transformations S such that for every vector x

$$Sx \in \text{null-Orb}(T)x$$

and we say that T is *algebraically null-orbit reflexive* if

$$\text{nullOrbRef}_0(T) = \text{nullOrb}(T).$$

If T is a bounded operator on a Banach space, we define $\text{nullOrbRef}(T)$ to be the set of all operators S such that, for every vector x

$$Sx \in [\text{nullOrb}(T)x]^-,$$

and we say that T is *null-orbit reflexive* if $\text{nullOrbRef}(T)$ is the strong-operator closure of $\text{nullOrb}(T)$. Orbit reflexivity is defined as in the above definition replacing $\text{nullOrb}(T)$ with $\text{Orb}(T)$. The slight change in definitions causes drastic changes in the two notions.

In this paper we extend all of the positive known results for orbit reflexivity to null-orbit reflexivity, and we show that most of the positive results for \mathbb{C} -orbit reflexivity extend to null orbit reflexivity. Moreover, for the example in [11] of a Hilbert space operator T , that is not orbit reflexive, we show that T is null-orbit reflexive. In the example in [1] of a Hilbert space operator that is not orbit reflexive, the proof shows that the operator is also not null-orbit reflexive.

We first prove a number of results in the purely algebraic case, and we use these to prove several results for operators on a normed space or a Hilbert space. We next extend the results of [5] and [11] to the null-orbit reflexivity case. We finish with a new result that every polynomially bounded operator on a Hilbert space is both orbit-reflexive and null-orbit reflexive.

Suppose X is a normed space and \mathcal{A} is an algebra of (bounded linear) operators on X . A (closed linear) subspace M of X is *\mathcal{A} -invariant* if $A(M) \subseteq M$ for every $A \in \mathcal{A}$. We let $\text{Lat}\mathcal{A}$ denote the set of all invariant subspaces for \mathcal{A} , and we let $\text{AlgLat}\mathcal{A}$ denote the algebra of all operators that leave invariant every \mathcal{A} -invariant subspace. The algebra \mathcal{A} is *reflexive* if $\mathcal{A} = \text{AlgLat}\mathcal{A}$. If the algebra \mathcal{A} contains the identity operator 1, then $S \in \text{AlgLat}\mathcal{A}$ if and only if, for every $x \in X$, Sx is in the closure of $\mathcal{A}x$. This characterization works equally well for a linear subspace \mathcal{S} of $B(X)$ (the set of all operators on X), i.e., we define $\text{ref}\mathcal{S}$ to be the set of all operators A such that, for every $x \in X$, we have Ax is in the closure of $\mathcal{S}x$, and we say that \mathcal{S} is reflexive if $\mathcal{S} = \text{ref}\mathcal{S}$. If we let T be a single operator and let $\mathcal{S} = \text{Orb}(T) = \{T^n : n \geq 0\}$, we apply the same process to obtain the notion of orbit reflexivity. (Note that in this case \mathcal{S} is not a linear space.) We define $\text{OrbRef}(T)$ to be the set of all operators A such that, for every vector x , we have Ax is in the closure of $\text{Orb}(T)x = \text{Orb}(T)x$. We say that T is *orbit reflexive* if $\text{OrbRef}(T)$ is the closure of $\text{Orb}(T)$ in the strong operator topology (SOT).

2 Algebraic Results

Throughout this section \mathbb{F} will denote an arbitrary field, X will denote a vector space over \mathbb{F} , and $\mathcal{L}(X)$ will denote the algebra of all linear transformations on X .

A transformation $T \in \mathcal{L}(X)$ is *locally nilpotent* if $X = \cup_{n \geq 1} \ker(T^n)$. More generally T is *locally algebraic* if, for each $x \in X$, there is a nonzero polynomial $p_x \in \mathbb{F}[t]$ such that $p_x(T)x = 0$. If $p_x(t)$ is chosen to be monic with minimal degree, we call p_x a *local polynomial* for T at x .

Theorem 1 *Every locally nilpotent linear transformation on a vector space X over field \mathbb{F} is algebraically null-orbit reflexive. Moreover, if $S \in \text{nullOrbRef}_0(T)$, $x \in X$, and $Sx = T^k x \neq 0$, then $S = T^k$.*

Proof. We know from [3, Theorem 1] that T is algebraically \mathbb{F} -orbit reflexive. Thus if $S \in \text{nullOrbRef}_0(T)$ and $S \neq 0$, then there is an $x \in X$ and an integer $n \geq 0$ such that $Sx = T^n x \neq 0$, and it follows from [3, Theorem 1] that $S = T^n$.

■

For infinite fields the next theorem reduces the problem of algebraic null-orbit reflexivity to the case of locally algebraic transformations. A key ingredient in the proof is an algebraic reflexivity result from [2] that says if \mathbb{F} is infinite and $T \in \mathcal{L}(X)$ is not locally algebraic, then, whenever $S \in \mathcal{L}(X)$ and for every $x \in X$ there is a polynomial p_x such that $Sx = p_x(T)x$, we must have $S = p(T)$ for some polynomial p .

Theorem 2 *Suppose X is a vector space over an infinite field \mathbb{F} , and suppose $T \in \mathcal{L}(X)$ is not locally algebraic. Then T is algebraically null-orbit reflexive.*

Proof. Suppose $S \in \text{nullOrbRef}_0(T)$. Then $Sx \in \text{nullOrb}(T)x$ for every $x \in X$. It follows from [2] that T is algebraically reflexive, so we know there is a polynomial $p \in \mathbb{F}[t]$ such that $S = p(T)$. Since T is not locally algebraic, there is a vector $e \in X$ such that for every nonzero polynomial $q \in \mathbb{F}[t]$, we have $q(T)e \neq 0$. Since $S \in \text{nullOrbRef}_0(T)$, we know that there is an $n \geq 0$ such that $Se = T^n e$. Hence $p(t) = t^n$, and thus $S \in \text{nullOrb}(T)$. ■

Remark 3 *If there is an $A \in \text{OrbRef}_0(T)$ such that $AT \neq TA$, then, since $\text{OrbRef}_0(T) \subseteq \text{nullOrbRef}_0(T)$, it follows that T is not algebraically null-orbit reflexive. Similarly, if T acts on a Banach space, and there is an $A \in \text{OrbRef}(T)$ such that $AT \neq TA$, then T is not null-orbit reflexive. Hence the Hilbert space operator constructed by S. Grivaux and M. Roginskaya [1] is not null-orbit reflexive.*

The preceding remark naturally leads to a pair of questions.

Question 1. If $S \in \text{nullOrbRef}_0(T)$ and $ST = TS$, must $S \in \text{nullOrb}(T)$?

Question 2. If T acts on a Hilbert space, $S \in \text{nullOrbRef}(T)$ and $ST = TS$, must S be in the strong-operator closure of $\text{nullOrb}(T)$? What is the answer if we assume that S is in the double commutant of $\{T\}$?

Note that the example of V. Müller and J. Vršovský [11, Example 1], where $S = 0 \in \text{OrbRef}(T) \setminus \text{Orb}(T)^{-\text{SOT}}$ shows that the analog of Question 2 for orbit reflexivity has a negative answer. We will see later (Corollary 15) that their example is null-orbit reflexive, so it has no bearing on Question 2. In [11] an example is given of an operator on ℓ^1 that is reflexive but not orbit reflexive. In view of Theorem 2.8 and Proposition 3.1 in [4], it seems feasible that the operator T in Example 1 of [11, Example 1] is reflexive. We know that $\text{AlgLat}T \subseteq \{T\}''$ and that if $S \in \text{AlgLat}T$, then there is a sequence $\{a_n\}_{n \geq 0}$

such that, for every vector x , $Sx \sim \sum_{n=0}^{\infty} a_n T^n$ in the sense of [4].

Question 3. Is the operator in Example 1 of [11, Example 1] is reflexive?

The proof of Theorem 2 shows that if T is algebraically \mathbb{F} -orbit reflexive (reflexive) and $\mathbb{F}\text{-Orb}(T)$ ($\{p(T) : p \in \mathbb{F}[t]\}$) has a separating vector, then T is algebraically null-orbit reflexive. This immediately gives us the following (see [3, Theorem 3]).

Theorem 4 *Suppose X is a finite-dimensional vector space over a field \mathbb{F} not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p . Then every linear transformation on X whose minimal polynomial splits over \mathbb{F} is algebraically null-orbit reflexive.*

Corollary 5 *If X is a finite-dimensional vector space over an algebraically closed field \mathbb{F} , then every linear transformation on X is algebraically null-orbit reflexive.*

Recall from ring theory that if \mathcal{R} is a principal ideal domain, M is an \mathcal{R} -module, $0 \neq r \in \mathcal{R}$ and $rM = \{0\}$, then M is a direct sum of cyclic \mathcal{R} -modules; Applying this fact to $\mathcal{R} = \mathbb{F}[t]$, we get that any algebraic linear transformation on a vector space is a direct sum of transformations on finite-dimensional subspaces, and therefore has a Jordan form when the minimal polynomial splits over \mathbb{F} . (See [6] for details.) This gives us the following corollary.

Corollary 6 *Suppose X is a vector space over a field \mathbb{F} not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p . Then every algebraic linear transformation on X whose minimal polynomial splits over \mathbb{F} is algebraically null-orbit reflexive.*

3 Null-orbit reflexivity

The following result was proved in [5, Proposition 3].

Lemma 7 *Suppose \mathcal{N} is a commuting family of normal operators on a Hilbert space X and $A \in B(X)$ satisfies, for every $x \in X$, $Ax \in (\mathcal{N}x)^\perp$. Then A is in the SOT-closure of \mathcal{N} .*

If in the preceding lemma we let $\mathcal{N} = \{0, 1, T, T^2, \dots\}$, we obtain the following.

Proposition 8 *Every normal operator on a Hilbert space is null-orbit reflexive.*

The next two results are consequences of Theorem 1.

Theorem 9 *Suppose T is a bounded linear operator on a real or complex normed space X such that $\cup_{n=1}^\infty \ker(T^n)$ is dense in X . Then T is null-orbit reflexive and $\text{nullOrb}(T)$ is SOT-closed. Moreover, if $S \in \text{nullOrbRef}(T)$, $x \in \cup_{n=1}^\infty \ker(T^n)$, $k \geq 0$, and $Sx = T^k x \neq 0$, then $S = T^k$.*

Proof. Suppose $S \in \text{nullOrbRef}(T)$, and let $M = \cup_{n=1}^\infty \ker(T^n)$. It is clear that $S(M) \subseteq M$ and $T(M) \subseteq M$ and $S|_M \in \text{nullOrbRef}_0(T|_M)$. But $T|_M$ is locally nilpotent, and if $x \in M$ and $T^n x = 0$, then

$$\text{nullOrb}(T)x = \{0\} \cup \{x, Tx, \dots, T^{n-1}x\}$$

is norm closed. Hence, $\text{nullOrbRef}(T|_M) = \text{nullOrbRef}_0(T|_M)$, which, by Theorem 1 is $\text{nullOrb}(T|_M)$. Hence there is an $A \in \text{nullOrb}(T)$ such that $S|_M = A|_M$. However, M is dense in X , so $S = A \in \text{nullOrb}(T)$. ■

The preceding theorem implies a stronger version of itself.

Corollary 10 *Suppose X is a real or complex normed space, and there is a decreasingly directed family $\{X_\lambda : \lambda \in \Lambda\}$ of T -invariant closed linear subspaces such that*

1. for every $\lambda \in \Lambda$, $\cup_{n=0}^\infty (T^n)^{-1}(X_\lambda)$ is dense in X , and
2. $\cap_{\lambda \in \Lambda} X_\lambda = \{0\}$.

Then T is null-orbit reflexive and $\text{nullOrbRef}(T) = \text{nullOrb}(T)$.

Proof. Suppose $S \in \text{nullOrbRef}(T)$ and $S \neq 0$. Choose $e \in X$ such that $Se \neq 0$. It follows from (2) that both (1) and (2) remain true if we consider only those X_λ that contain neither e nor Se . Since $T(X_\lambda) \subseteq X_\lambda$, $\hat{T}_\lambda(x + X_\lambda) = Tx + X_\lambda$ defines a bounded linear operator \hat{T}_λ on X/X_λ . Condition (1) implies that

$\cup_{n=1}^{\infty} \ker(\hat{T}_{\lambda}^n)$ is dense in X/X_{λ} ; whence, by Theorem 9, \hat{T}_{λ} is null-orbit reflexive. However, $S \in \text{nullOrbRef}(T)$ implies that $S(X_{\lambda}) \subseteq X_{\lambda}$, so $\hat{S}_{\lambda}(x + X_{\lambda}) = Sx + X_{\lambda}$ defines an operator on X/X_{λ} such that $\hat{S}_{\lambda} \in \text{nullOrbRef}(\hat{T}_{\lambda})$. Hence, by Theorem 9, there is a unique nonnegative integer n_{λ} such that $\hat{S}_{\lambda} = \hat{T}_{\lambda}^{n_{\lambda}}$. Suppose $\eta \in \Lambda$. Since the X_{λ} 's are decreasingly directed, there is a $\sigma \in \Lambda$ such that $X_{\sigma} \subseteq X_{\lambda} \cap X_{\eta}$. Applying the same arguments we used on X_{λ} , there is a unique integer $m \geq 0$ such that $\hat{S}_{\sigma} = T_{\sigma}^{n_{\sigma}}$. However, it follows from (1) that there is a vector $x \in \left[\cup_{n=0}^{\infty} (T^n)^{-1}(X_{\sigma})\right] \setminus X_{\lambda}$. Then there is an n such that $T^n x \in X_{\sigma} \subseteq X_{\lambda}$ and thus $\hat{T}^n(x + X_{\lambda}) = 0$ but $x + X_{\lambda} \neq 0$. However, $Sx - T^{n_{\sigma}}x \in X_{\sigma} \subseteq X_{\lambda}$, so

$$\hat{S}_{\lambda}(x + X_{\lambda}) = \bar{T}_{\lambda}^{n_{\sigma}}(x + X_{\lambda}) = \bar{T}_{\lambda}^{n_{\lambda}}(x + X_{\lambda}),$$

which implies that $n_{\sigma} = n_{\lambda}$. Similarly, $n_{\sigma} = n_{\lambda}$. Hence there is an integer $n \geq 0$ such that, for every $\lambda \in \Lambda$, $n_{\lambda} = n$. Hence, for every $x \in X$ and every $\lambda \in \Lambda$,

$$Sx - T^n x \in X_{\lambda},$$

which, by (2), implies $S = T^n$. ■

The following corollary applies to operators that have a strictly upper-triangular operator matrix with respect to some direct sum decomposition.

Corollary 11 *If a normed space X over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is a direct sum of spaces $\{X_n : n \in \mathbb{N}\}$ such that $T(X_1) = \{0\}$, and for every $n > 1$,*

$$T(X_n) \subseteq \left(\sum_{k < n}^{\oplus} X_k\right)^{-},$$

then T is null-orbit reflexive and $\text{nullOrbRef}(T) = \text{nullOrb}(T)$.

The preceding corollary has some familiar special cases.

Corollary 12 *If T is an operator-weighted (unilateral, bilateral, or backwards) shift or if T is a direct sum of nilpotent operators on a real or complex normed space X , then T is null-orbit reflexive.*

Theorem 13 *Suppose X is a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $T \in B(X)$ and $\cap_{n=1}^{\infty} T^n(X)^{-} = \{0\}$. Then T is null-orbit reflexive and $\text{nullOrbRef}(T) = \text{nullOrb}(T)$. Moreover, if $S \in \text{nullOrbRef}(T)$, $x \in X$, and $0 \neq Sx = T^k x$, then $S = T^k$.*

Proof. We will first show that T is algebraically null-orbit reflexive. If M is a finite-dimensional invariant subspace for T and $T|M$ is not nilpotent, then there is a nonzero T -invariant subspace N of M such that $\ker(T|N) = 0$. Thus $T(N) = N \neq 0$, which violates $\cap_{n=1}^{\infty} T^n(X)^- = \{0\}$. Thus, either T is not locally algebraic or T is locally nilpotent. In these cases it follows either from Theorem 2 or Theorem 1 that T is indeed algebraically null-orbit reflexive. Furthermore, the hypothesis on T implies, for each $x \in X$, that

$$\cap_{N=1}^{\infty} \{T^k x : k \geq N\}^- = \{0\},$$

so $\text{nullOrb}(T)x$ is closed in X . Thus $\text{nullOrbRef}(T) = \text{nullOrbRef}_0(T) = \text{nullOrb}(T)$. For the last statement suppose $x \in X$, and $k, n \geq 0$ are integers, and

$$0 \neq Sx = T^n x = T^k x.$$

Suppose $k < n$. Then $M = \text{sp}\{x, Tx, \dots, T^{n-1}x\}$ is a nonzero finite-dimensional invariant subspace for T with $\dim M \leq n$. Since $T^n x \neq 0$, we know $T|M$ is not nilpotent, which, as remarked earlier, contradicts $\cap_{n=1}^{\infty} T^n(X)^- = \{0\}$. ■

This theorem also implies a stronger version of itself.

Corollary 14 *Suppose X is a real or complex normed space, $T \in B(X)$, and there is an increasingly directed family $\{X_\lambda : \lambda \in \Lambda\}$ of T -invariant linear subspaces such that*

1. for every $\lambda \in \Lambda$, $\cap_{n=1}^{\infty} \overline{T^n(X_\lambda)} = \{0\}$, and
2. $\cup_{\lambda \in \Lambda} X_\lambda$ is dense in X .

Then T is null-orbit reflexive, and $\text{nullOrbRef}(T) = \text{nullOrb}(T)$. Moreover, if $S \in \text{nullOrbRef}(T)$, $x \in X$, and $0 \neq Sx = T^k x$, then $S = T^k$.

Proof. Suppose $0 \neq S \in \text{nullOrbRef}(T)$. It follows from (2) that there is a $\lambda_0 \in \Lambda$ and an $f \in X_{\lambda_0}$ such that $0 \neq Sf$. However, we must have $S(X_{\lambda_0}) \subseteq X_{\lambda_0}$, and $S|X_{\lambda_0} \in \text{nullOrbRef}(T|X_{\lambda_0}) = \text{nullOrb}(T|X_{\lambda_0})$ (by (1) and the preceding theorem). Thus there is an integer $k \geq 0$ such that

$$S|X_{\lambda_0} = T^k|X_{\lambda_0}.$$

The same k must work for any X_λ that contains X_{λ_0} . It follows from the fact that the family is increasingly directed and (2) that $S = T^k$. ■

If T is the operator constructed in [11] that is not orbit reflexive, it is easy to show that $\cap_{n \geq 0} T^n(X)^- = 0$.

Corollary 15 *The non orbit reflexive operator constructed in [11] is null-orbit reflexive.*

Irving Kaplansky [6] (see also [7], [8], [10]) proved that a (bounded linear) operator on a Banach space is locally algebraic if and only if it is algebraic. This immediately gives us the following result from Theorem 2.

Proposition 16 *Suppose X is a real or complex Banach space and $T \in B(X)$ is not algebraic. Then T is algebraically null-orbit reflexive.*

The results in the paper of [11] also extend to the null-orbit case. If T is an operator on a Banach space, then $r(T)$ denotes the spectral radius of T , i.e.,

$$r(T) = \max \{ |\lambda| : \lambda \in \sigma(T) \}.$$

Lemma 17 *If X is a normed space, $T \in B(X)$ and*

$$E = \{x \in X : \text{nullOrb}(T)x \text{ is norm closed}\}$$

is not contained in a countable union of nowhere dense subsets of X , then T is null-orbit reflexive and $\text{nullOrbRef}(T) = \text{nullOrb}(T)$. (Note that E contains all $x \in X$ such that $T^n x \rightarrow 0$ weakly or $\|T^n x\| \rightarrow \infty$.)

Proof. If $S \in \text{nullOrbRef}(T)$, then $E \subseteq \bigcup_{A \in \text{nullOrb}(T)} \ker(S - A)$, so there is an $A \in \text{nullOrb}(T)$ such that $\ker(S - A)$ has nonempty interior, which means that $S = A$. ■

Corollary 18 *If X is a Banach space, $T \in B(X)$ and $r(T) < 1$, then T is null-orbit reflexive.*

Proof. It follows that $\|T^n\| \rightarrow 0$, and thus the set E in Lemma 17 is all of X .

■

The proof of the following theorem is almost exactly the same as the proof of Theorem 7 in [11].

Theorem 19 *If X is a Banach space and $T \in B(X)$ and $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$, then T is null-orbit reflexive. If X is a Hilbert space and $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^2} < \infty$, then T is null-orbit reflexive. In particular, if $r(T) \neq 1$, then T is null-orbit reflexive.*

Corollary 20 *The set of null-orbit reflexive operators on a Banach space X is norm dense in $B(X)$.*

Theorem 21 *If X is a Hilbert space and $T \in B(X)$ and is polynomially bounded, then T is null-orbit reflexive and orbit reflexive.*

Proof. We prove the null-orbit reflexivity; the orbit reflexivity is proved in a similar fashion. Suppose T is polynomially bounded. It was proved by W. Mlak [9] that T is similar to the direct sum of a unitary operator U and an operator A with a weakly continuous H^∞ functional calculus. In particular, $A^n \rightarrow 0$ in the weak operator topology. We can assume $T = U \oplus A$. We can also assume that the A summand is present; otherwise, T is null-orbit reflexive by Proposition 8. Since $A^n \rightarrow 0$ in WOT, it follows from Lemma 17 that $\text{nullOrbRef}(A) = \text{nullOrb}(A)$. Hence we can assume that the U summand is also present. Suppose $S \in \text{nullOrbRef}(T)$. Then we can write $S = B \oplus C$. Hence $C \in \text{nullOrb}(A)$. We also know that $B \in \text{nullOrbRef}(U)$.

Case 1. $C = 0$, and $B \neq 0$. For a fixed x_0 with $Bx_0 \neq 0$ and any y there is a sequence $\{n_k\}$ of nonnegative integers such that $\|T^{n_k}(x_0 \oplus y) - Bx_0 \oplus 0\| \rightarrow 0$. In particular, $\|A^{n_k}y\| \rightarrow 0$. However, $A^n \rightarrow 0$ WOT implies there is an $M > 0$ such that $\|A^n\| < M$ for all $n \geq 0$. We want to show $\|A^n y\| \rightarrow 0$. Suppose $\varepsilon > 0$. Then there is an n_k such that $\|A^{n_k}y\| < \varepsilon/M$. If $n \geq n_k$, then

$$\|A^n y\| \leq \|A^{n-n_k}\| \|A^{n_k}y\| < M(\varepsilon/M) = \varepsilon.$$

We now know that $A^n \rightarrow 0$ in the strong operator topology.

Now suppose $m \geq 0$ and $A^m \neq 0$. Choose y_0 such that $A^m y_0 \neq 0$. For any x , there is a sequence $\{n_k\}$ of integers such that $T^{n_k}(x \oplus y_0) \rightarrow S(x \oplus y_0)$, and it follows that eventually $n_k > m$. Thus, for every x we have $Bx \in \{U^n x : n > m\}$, so it follows from Lemma 17 that $B \in \{U^n : n > m\}^{-SOT}$. It now follows that there is a net $\{n_\lambda\}$ of positive integers such that $T^{n_\lambda} \rightarrow S$ in the strong operator topology.

Case 2. $C \neq 0$. Since $C \in \text{nullOrb}(A)$, there is an integer $s \geq 0$ such that $C = A^s \neq 0$. Since $A^n \rightarrow 0$ in the WOT, it follows that $\text{Ker}(A^k - 1) = 0$ for $k > 0$. Thus if $A^n y = A^m y$ with $n < m$, then $(A^{m-n} - 1)A^n y = 0$, which implies that $A^n x = 0$ and therefore $A^m x = 0$. Choose y_1 so that $A^s y_1 \neq 0$. It follows that if $\{n_k\}$ is a sequence of nonnegative integers and $A^{n_k} y_1 \rightarrow A^s y_1$, then n_k must eventually become s . By considering vectors of the form $x \oplus y_1$, we see that $B = U^s$, and therefore $S = T^s$.

Since the only remaining case is $S = 0 \in \text{nullOrb}(T)$, the proof is complete.

■

Corollary 22 *If T is a Hilbert space operator and $\|T\| \leq 1$, then T is null-orbit reflexive.*

Corollary 23 *If T is a Hilbert space operator with $\|T\| = r(T)$ (e.g., T is hyponormal), then T is null-orbit reflexive.*

The following lemma is a consequence of Theorem 19.

Lemma 24 *Suppose X is a Hilbert space, $T \in B(X)$, $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. If $\ker(T - \lambda) \neq \ker(T - \lambda)^2$, then T is null orbit reflexive.*

Proof. Suppose $\|x\| = 1$ and $(T - \lambda)^2 x = 0$ and $(T - \lambda)x \neq 0$. It follows that

$$\|T^n x\| = \|[\lambda + (T - \lambda)]^n x\| = \|\lambda^n x + n(T - \lambda)x\| \geq n\|(T - \lambda)x\| - \|x\| \geq \varepsilon n$$

for some $\varepsilon > 0$ and for sufficiently large n . Thus $\sum 1/\|T^n\|^2 < \infty$, which, by Theorem 19, implies T is null-orbit reflexive. ■

Theorem 25 *Suppose X is a Hilbert space, $T \in B(X)$, $r(T) = 1$ and no point in $E = \sigma(T) \cap \{z \in \mathbb{C} : |z| = 1\}$ is a limit point of the spectrum of T . If the restriction of T to the spectral subspace M_E for the clopen subset E of $\sigma(T)$ is an algebraic operator, then T is null-orbit reflexive. In particular, every compact operator, or algebraic operator on a Hilbert space is null-orbit reflexive. Hence every operator on a finite-dimensional space is null-orbit reflexive.*

Proof. It follows from Lemma 24 that we need only consider the case when $\ker(T - \lambda) = \ker(T - \lambda)^2$ for every $\lambda \in E$. This implies that the restriction of T to M_E is similar to a unitary operator, and since the restriction of T to $M_{\sigma(T) \setminus E}$ has spectral radius less than 1, we see that T is similar to a contraction. Hence, by Theorem 21, T is null-orbit reflexive. If T is compact or algebraic and $r(T) = 1$, then the first part of this theorem applies. If $r(T) \neq 1$, then T is null-orbit reflexive by Theorem 19. ■

We conclude with another question.

Question 4. Is every power bounded Hilbert space operator orbit reflexive or null-orbit reflexive?

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